The Preface Paradox (and a Probability Primer)

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Norms of Rational Belief

What norms govern rational belief? Here are two plausible candidates.

In order for your beliefs to responsibly represent the world, they should be *consistent*. Otherwise, it's not logically possible for them all to be true!

Belief Consistency: Rationality requires the set of things you believe the be logically consistent.

$$\Rightarrow \neg B(p \wedge \neg p)$$

In order for your beliefs to responsibly represent the world, they should be *closed under logical entailment*. If X logically entails Y, then Y must be true if X is. And so, if you believe X is true, you should believe Y is true, too.

Belief Closure: If some of your beliefs entail a further proposition, rationality requires you to believe that further proposition as well.

$$\Rightarrow \text{ If } B(p) \wedge B(q), \text{ then } B(p \wedge q)$$

Are these plausible rules? What can be said to justify them?

The Preface Paradox

Consider the following example from Makinson (1965):

> You write a long, painstakingly-researched work of nonfiction, which contains many claims in its main text, each of which you believe. In the preface at the beginning of the book you write:
>
> > "I am indebted to many for their invaluable help and encouragement. I am sure there remain errors in the main text, for which I take sole responsibility."
>
> Everybody makes mistakes. And you, being a reasonable fellow, recognize that you are no exception.

Let me highlight two features of this story.

Let $p_1, p_2, \ldots, p_n$ stand for all of the claims you make in the book.

Note: $(\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$ is logically equivalent to $\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n)$

Rational Humility: It's rational to believe that at least one of the claims in your book is false.

$$\Rightarrow B(\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n))$$

Honesty: Every claim you make in the main text of your book you believe to be true.

$$\Rightarrow B(p_1) \wedge B(p_2) \wedge \cdots \wedge B(p_n)$$

Consistency, Closure, Humility, and Honesty can't all be true! Which should we give up?

Idea: Binary beliefs → degrees of belief ('credences')

The Preface Paradox

(1)	$B(p_1 \wedge p_2 \wedge \cdots \wedge p_n)$	[Honesty, Closure]
(2)	Let $p = (p_1 \wedge p_2 \wedge \cdots \wedge p_n)$	
(3)	$B(p)$	[(1), (2)]
(4)	$B(\neg p)$	[Humility, (2)]
(5)	$B(p \wedge \neg p)$	[(3), (4), Closure]
(6)	$\neg B(p \wedge \neg p)$	[Consistency]
(7)	$\bot$	[(5), (6)]

Degrees of Belief and Probabilism

What norms govern rational *degrees* of belief? Here's a plausible idea:

Probabilism: Rationality requires your degrees of belief to obey the probability axioms.

Before investigating whether this is true, let's unpack what this says.

The Probability Axioms

NON-NEGATIVITY. Every $X \in \mathscr{L}$ is assigned a non-negative number.

$$C(X) \geq 0 \quad (1)$$

NORMALITY. Every tautology $\top \in \mathscr{L}$ is assigned 1.

$$C(\top) = 1 \quad (2)$$

FINITE ADDITIVITY. For any mutually exclusive $X, Y \in \mathscr{L}$, the number assigned to their disjunction equals the sum of the numbers assigned to them.

$$\text{If } (X \wedge Y) \vdash \bot, \text{ then } c(X \vee Y) = c(X) + c(Y) \quad (3)$$

Here are three interesting and useful facts.

The Negation Rule: For any $X \in \mathscr{L}$, $c(\neg X) = 1 - c(X)$.

The Overlap Rule: In general, the probability of a disjunction equals the sum of the probabilities of its disjuncts minus the probability of its disjuncts' overlap.

$$c(X \vee Y) = c(X) + c(Y) - c(X \wedge Y)$$

The Logical Consequence Rule: If $X \vdash Y$, then $c(X) \leq c(Y)$.

The Conjunction Fallacy. In a famous study, Tversky and Kahneman (1983) presented subjects with the following story:

> Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

The subjects were then asked to rank the probabilities of the following propositions:

- Linda is active in the feminist movement.
- Linda is a bank teller.
- Linda is a bank teller and is active in the feminist movement.

A large majority of the subjects ranked the third option as more probable than the second!

Conditional Probability

In addition to the three axioms above, we introduce the notion of *conditional probability*.

$$c(X \mid Y) = \frac{c(X \wedge Y)}{c(Y)} \quad (4)$$

This tells us the probability of X being the case *conditional* on Y being the case. Here's another useful definition:

Independence: X and Y are statistically independent just in case $c(X \mid Y) = c(X)$.